Joining of Complex Substructures by the Matrix Force Method

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A new method of matrix force analysis is described for the calculation of stresses and deflections in complex structures partitioned into a number of structural components (substructures). Each substructure first is analyzed separately assuming that all common joints with the adjacent substructures are disconnected and the joint internal forces on the common boundaries are replaced by the substructure interaction forces. The magnitudes of the interaction forces are determined from the external and internal equilibrium equations and the displacement compatibility equations. The selection of the redundant interaction forces for the compatibility equations is carried out automatically using Jordanian elimination technique. Once the magnitudes of the interaction forces are known, the internal forces (or stresses) and deflections are then determined from the analysis of individual substructures. All steps in the analysis are suitable for a fully automatic computer program. A numerical example is included to show the details of the analysis.

Nomenclature

$\mathbf{F}^{(r)}$ = interaction forces, rth substructure
$\mathbf{Q}^{(r)}$ = substructure reaction forces, rth substructure
$\mathbf{\Phi}^{(r)}$ = external loads, rth substructure
$\mathbf{F}$ = $\{\mathbf{F}^{(1)}\mathbf{F}^{(2)}, \ldots, \mathbf{F}^{(N)}\}$, interaction forces
$\mathbf{Q}$ = $\{\mathbf{Q}^{(1)}\mathbf{Q}^{(2)}, \ldots, \mathbf{Q}^{(N)}\}$, substructure reaction forces
$\mathbf{\Phi}$ = $\{\mathbf{\Phi}^{(1)}\mathbf{\Phi}^{(2)}, \ldots, \mathbf{\Phi}^{(N)}\}$, external loads
$\mathbf{R}$ = reaction force complements on the joined structure; forces applied by the structure to the supports
$\bar{\mathbf{F}}$ = $\{\mathbf{R}\,\mathbf{F}\}$, substructure boundary forces
$\bar{\mathbf{F}}_0$ = substructure boundary forces in the statically determinate cut structure
$\mathbf{X}$ = interaction redundancies
$\bar{\mathbf{F}}_\Phi$ = substructure boundary forces in the joined structure due to unit values of $\mathbf{\Phi}$
$\bar{\mathbf{F}}_e$ = substructure boundary forces in the joined structure due to unit values of $\mathbf{e}$
$\mathbf{X}_\Phi$ = interaction redundancies due to unit values of $\mathbf{\Phi}$
$\mathbf{X}_e$ = interaction redundancies due to unit values of $\mathbf{e}$
$\mathbf{Q}_F{}^{(r)}$ = substructure reaction forces due to unit values of $\mathbf{F}^{(r)}$, rth substructure
$\mathbf{Q}_\Phi{}^{(r)}$ = substructure reaction forces due to unit values of $\mathbf{\Phi}^{(r)}$, rth substructure
$\mathbf{Q}_F$ = substructure reaction forces due to unit values of interaction forces $\mathbf{F}$
$\mathbf{Q}_\Phi$ = substructure reaction forces due to unit values of external loads $\mathbf{\Phi}$
$\mathbf{P}_Q\ \mathbf{P}_R\ \mathbf{P}_F$ = submatrices in the internal equilibrium equations, Eq. (10)
$\bar{\mathbf{P}}_F$ = matrix defined by Eq. (12)
$\bar{\mathbf{P}}_\Phi$ = matrix defined by Eq. (13)
$\bar{\mathbf{P}}$ = $[\mathbf{P}_R\,\bar{\mathbf{P}}_F]$
$\mathbf{M}$ = matrix defined by Eq. (19)
$\bar{\mathbf{P}}_x$ = matrix defined by Eq. (19)
$\mathbf{q}_x$ = $-\mathbf{M}^{-1}\bar{\mathbf{P}}_x$
$\mathbf{q}_\Phi$ = $-\mathbf{M}^{-1}\bar{\mathbf{P}}_\Phi$
$\mathbf{f}_x$ = substructure boundary forces in the cut structure due to unit values of $\mathbf{X}$
$\mathbf{f}_\Phi$ = substructure boundary forces in the cut structure due to unit values of $\mathbf{\Phi}$
$\mathbf{f}_\Delta$ = substructure boundary forces in the cut structure due to unit values of external loads in the $\mathbf{\Delta}$ directions
$\mathbf{D}_{xx}$ = $\mathbf{f}_x{}^T\bar{\mathbf{D}}_{FF}\mathbf{f}_x$
$\mathbf{D}_{x\Phi}$ = $\mathbf{f}_x{}^T\bar{\mathbf{D}}_{FF}\mathbf{f}_\Phi + \mathbf{f}_x{}^T\bar{\mathbf{D}}_{F\Phi}$
$\mathbf{D}_{xe}$ = $\mathbf{f}_x{}^T\mathbf{e}$
$\mathbf{v}_F{}^{(r)}$ = rth substructure displacements in the directions of $\mathbf{F}^{(r)}$, relative to substructure datum; unassembled structure
$\mathbf{v}_F$ = $\{\mathbf{v}_F{}^{(1)}\ \mathbf{v}_F{}^{(2)}, \ldots, \mathbf{v}_F{}^{(N)}\}$
$\bar{\mathbf{v}}_F$ = $\{\mathbf{e}_R\ \mathbf{v}_F\}$
$\mathbf{v}_\Delta$ = relative displacements in the directions of $\mathbf{\Delta}$
$\mathbf{e}_F{}^{(r)}$ = rth substructure initial displacements in the directions of $\mathbf{F}^{(r)}$, relative to substructure datum; unassembled structure
$\mathbf{e}_F$ = $\{\mathbf{e}_F{}^{(1)}\ \mathbf{e}_F{}^{(2)}, \ldots, \mathbf{e}_F{}^{(N)}\}$
$\mathbf{e}_R$ = external reactions displacements, e.g., sinking of the supports
$\mathbf{e}$ = $\{\mathbf{e}_R\ \mathbf{e}_F\}$
$\mathbf{\Delta}$ = joined structure displacements
$\mathbf{\Delta}_\Phi$ = joined structure displacements due to unit values of $\mathbf{\Phi}$
$\mathbf{\Delta}_e$ = joined structure displacements due to unit values of $\mathbf{e}$
$\mathbf{D}_{FF}{}^{(r)}$ = displacements in the directions of $\mathbf{F}^{(r)}$ due to unit values of $\mathbf{F}^{(r)}$
$\mathbf{D}_{F\Phi}{}^{(r)}$ = displacements in the directions of $\mathbf{F}^{(r)}$ due to unit values of $\mathbf{\Phi}^{(r)}$
$\mathbf{D}_{\Phi F}{}^{(r)}$ = displacements in the directions of $\mathbf{\Phi}^{(r)}$ due to unit values of $\mathbf{F}^{(r)}$
$\mathbf{D}_{\Phi\Phi}{}^{(r)}$ = displacements in the directions of $\mathbf{\Phi}^{(r)}$ due to unit values of $\mathbf{\Phi}^{(r)}$
$\mathbf{D}^{(r)}$ = rth substructure flexibility matrix, relative to substructure datum
$\mathbf{D}_{FF}$ = $\lceil \mathbf{D}_{FF}{}^{(1)}\ \mathbf{D}_{FF}{}^{(2)}, \ldots, \mathbf{D}_{FF}{}^{(N)} \rfloor$
$\mathbf{D}_{F\Phi}$ = $\lceil \mathbf{D}_{F\Phi}{}^{(1)}\ \mathbf{D}_{F\Phi}{}^{(2)}, \ldots, \mathbf{D}_{F\Phi}{}^{(N)} \rfloor$
$\mathbf{D}_{\Phi\Phi}$ = $\lceil \mathbf{D}_{\Phi\Phi}{}^{(1)}\ \mathbf{D}_{\Phi\Phi}{}^{(2)}, \ldots, \mathbf{D}_{\Phi\Phi}{}^{(N)} \rfloor$
$\bar{\mathbf{D}}_{FF}$ = displacements in the directions of $\bar{\mathbf{F}}$ due to unit values of $\bar{\mathbf{F}}$
$\bar{\mathbf{D}}_{F\Phi}$ = displacements in the directions of $\bar{\mathbf{F}}$ due to unit values of $\mathbf{\Phi}$
$\bar{\mathbf{D}}_{\Phi F}$ = displacements in the directions of $\mathbf{\Phi}$ due to unit values of $\bar{\mathbf{F}}$
$\bar{\mathbf{D}}_{\Delta F}$ = displacements in the directions of $\mathbf{\Delta}$ due to unit values of $\bar{\mathbf{F}}$
$\mathbf{D}_{\Delta\Phi}$ = displacements in the directions of $\mathbf{\Delta}$ due to unit values of $\mathbf{\Phi}$
$\mathbf{N}$ = extractor matrix defined by Eq. (28)
$\mathbf{H}_x$ = extractor matrix defined by Eq. (29)
$\bar{\mathbf{C}}_\Delta$ = column extractor matrix defined by Eq. (64)
$\mathbf{C}_\Delta$ = row extractor matrix defined by Eqs. (65) and (66)
N = number of substructures
[ ] = matrix
⌈ ⌋ = diagonal matrix
{ } = column matrix

Received June 23, 1965; revision received January 11, 1966. The paper was prepared while Professor Przemieniecki was with Douglas Aircraft Company on leave of absence from the Air Force. The work described in this paper was supported by the Independent Research and Development Program of Douglas Aircraft Company, Inc.

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Introduction

IN applying matrix methods of analysis to large structures, the number of structural elements very often exceeds the capacity of available computer programs, and consequently some form of structural partitioning must be employed. Structural partitioning involves division of the complete structure into a number of substructures, the boundaries of which may be specified arbitrarily; however, for convenience it is preferable to make structural partitioning corresponding to physical partitioning. If the stiffness or flexibility properties of each substructure are determined, then the substructures may be treated as complex structural elements and the matrix displacement or force methods of structural analysis can be formulated for the partitioned structure. Once the displacements or forces on substructure boundaries have been found, each substructure then can be analyzed separately under known substructure boundary displacements or forces, depending on whether displacement or force methods of analysis are used. The substructure analysis involves a relatively small structure for which the available computer programs can be used.

In a recent paper Przemieniecki[1] developed a displacement method of analysis for partitioned structures. In this method each substructure is analyzed first with all displacements on the common boundaries between adjacent substructures completely constrained. These boundaries then are relaxed subsequently and their displacements are determined from the equations of equilibrium of the boundary forces. The substructures are analyzed separately then, again under the action of specified external loading and the previously determined boundary displacements. Turner, Martin, and Weikel[2] considered the analysis of partitioned structures also by the displacement method in which the substructures are treated as complex elements. Argyris and Kelsey[3] analyzed partitioned structures using the matrix force method in which the compatibility of deformations between adjacent substructures is ensured by the redundant interaction force systems.

The object of this paper is to present the elastic analysis of large structures using the substructure concept with the boundary interaction forces as unknowns in the analysis. Thus the present method of analysis may be placed in the category of matrix force methods. Although the method is somewhat similar to the method of Ref. 3, it differs, however, in one very important aspect. Argyris and Kelsey formulate manually their redundant interaction systems, which has the obvious disadvantage that different systems must be prepared for different structural configurations, whereas in the present method the formulation of the redundant interaction systems is carried out automatically in the computer. The automatic selection of redundant force systems, developed originally by Denke and his associates,[4] gives the present method a great flexibility in practical applications since it can be applied to any structural configuration. The method of analysis described in this paper has been programed for IBM 7094 digital computer. The program was used successfully for the analysis of large structures for which, because of their size, other available computer programs could not be utilized.

Unassembled Substructures

It will be assumed that the complete structure is divided into N substructures. A typical substructure arrangement is shown in Fig. 1 where a medium range transport aircraft structure is divided into six structural components: wing, center fuselage, front fuselage, rear fuselage, engine pylon, and vertical stabilizer. Now consider a typical substructure that has been isolated from the remaining substructures (Fig. 2). The external loading applied to this particular substructure (r) will be denoted by a column matrix $\mathbf{\Phi}^{(r)}$, where the superscript r denotes the r-th substructure. Since

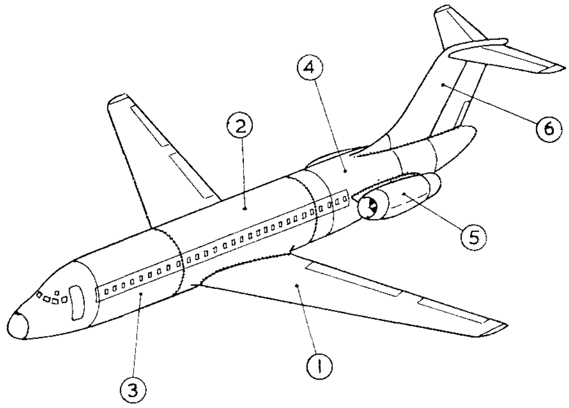

Fig. 1 Typical substructure arrangement.

the disconnecting procedure is carried out by cutting all attachments on the boundaries, we therefore must restore equilibrium and compatibility of unassembled (disconnected) substructures by applying joint internal forces to each substructure. For the r-th substructure, these forces will be denoted by column matrices $\mathbf{Q}^{(r)}$ and $\mathbf{F}^{(r)}$, with $\mathbf{Q}^{(r)}$ representing substructure reactions introduced only temporarily to establish a reference datum for substructure displacements and flexibilities, and $\mathbf{F}^{(r)}$ representing all the remaining boundary forces. The forces $\mathbf{F}^{(r)}$ will be referred to as the interaction forces. The only restriction imposed on $\mathbf{Q}^{(r)}$ is that it must constitute a set of statically determinate reactions capable of reacting any general loading. Thus for a general three-dimensional structure there will be six forces in $\mathbf{Q}^{(r)}$.

The equation of external equilibrium for the r-th substructure can be written in matrix form as

$$\mathbf{Q}^{(r)} = [\mathbf{Q}_F{}^{(r)}\mathbf{Q}_\Phi{}^{(r)}]\begin{bmatrix}\mathbf{F}^{(r)}\\ \mathbf{\Phi}^{(r)}\end{bmatrix} \tag{1}$$

where $\mathbf{Q}_F{}^{(r)}$ and $\mathbf{Q}_\Phi{}^{(r)}$ denote substructure reaction forces due to unit values of $\mathbf{F}^{(r)}$ and $\mathbf{\Phi}^{(r)}$, respectively. The formulation of Eq. (1) involves only equations of statics, since the substructure reactions $\mathbf{Q}^{(r)}$ are determinate statically. Eq. (1) for all substructures can now be assembled into a single matrix equation

$$\mathbf{Q} = [\mathbf{Q}_F\ \mathbf{Q}_\Phi]\begin{bmatrix}\mathbf{F}\\ \mathbf{\Phi}\end{bmatrix} \tag{2}$$

where

$$\mathbf{Q} = \{\mathbf{Q}^{(1)}\ \mathbf{Q}^{(2)}, \ldots, \mathbf{Q}^{(N)}\} \tag{3}$$

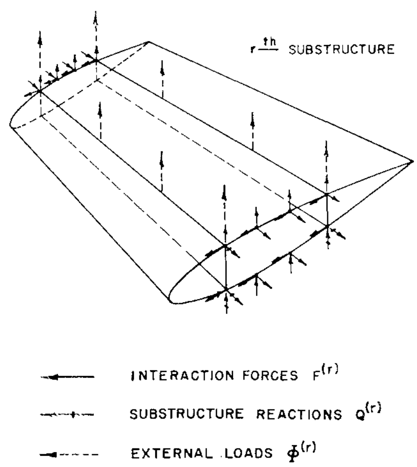

Fig. 2 Substructure forces.

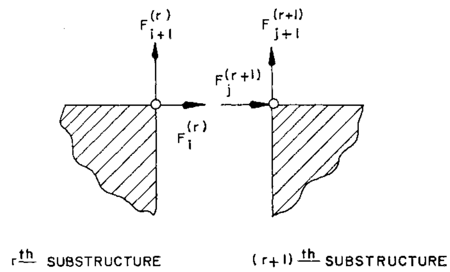

Fig. 3 Interaction forces at a typical joint.

$$\mathbf{Q}_F = [\mathbf{Q}_F^{(1)}\ \mathbf{Q}_F^{(2)}, \ldots, \mathbf{Q}_F^{(N)}] \tag{4}$$

$$\mathbf{Q}_\Phi = [\mathbf{Q}_\Phi^{(1)}\ \mathbf{Q}_\Phi^{(2)}, \ldots, \mathbf{Q}_\Phi^{(N)}] \tag{5}$$

$$\mathbf{F} = \{\mathbf{F}^{(1)}\ \mathbf{F}^{(2)}, \ldots, \mathbf{F}^{(N)}\} \tag{6}$$

$$\mathbf{\Phi} = \{\mathbf{\Phi}^{(1)}\ \mathbf{\Phi}^{(2)}, \ldots, \mathbf{\Phi}^{(N)}\} \tag{7}$$

Internal Equilibrium of the Joined Structure

When the substructures are joined together the external loading $\mathbf{\Phi}$ is reacted by the joined structure reactions. The force complements to these reactions will be denoted by the column matrix $\mathbf{R}$. The reaction force complement is defined here as the force applied by the structure to the support. The minimum number of reactions is equal to the number of rigid body degrees of freedom for the structure; however, additional reactions may be needed to represent redundant constraints. For example, if only one half of a symmetric structure is analyzed under a symmetric loading, then reactions across the plane of symmetry must be introduced to represent the symmetry constraint. The interaction forces $\mathbf{F}$, substructure reactions $\mathbf{Q}$, and the joined structure reaction complements $\mathbf{R}$ must be in equilibrium at each joint on the substructure boundaries. A typical joint is shown in Fig. 3 for which the equations of internal equilibrium are

$$F_i^{(r)} + F_j^{(r+1)} = 0 \tag{8}$$

$$F_{i+1}^{(r)} + F_{j+1}^{(r+1)} = 0 \tag{9}$$

Naturally at some joints the equilibrium equations may contain the substructure reaction force $\mathbf{Q}$ or the joined structure reaction force complement $\mathbf{R}$.

Equations of internal equilibrium may be formulated with reference to a common set of axes. Generally, however, it is preferable to use pairs of corresponding boundary forces and formulate equilibrium equations in the direction of each pair. The internal equilibrium requires then that the sum of the two forces in every pair must be equal to zero.

The equations of internal equilibrium at the boundary joints therefore can be obtained simply from the substructure connectivity. These equations can be expressed in the form

$$[\mathbf{P}_Q\ \mathbf{P}_R\ \mathbf{P}_F]\begin{bmatrix}\mathbf{Q}\\ \mathbf{R}\\ \mathbf{F}\end{bmatrix} = 0 \tag{10}$$

where every row in the submatrices $\mathbf{P}_Q$, $\mathbf{P}_R$, and $\mathbf{P}_F$ contains only zeros and ones. Substituting Eq. (2) into (10) to eliminate $\mathbf{Q}$, we have

$$[\mathbf{P}_R\ \bar{\mathbf{P}}_F\ \bar{\mathbf{P}}_\Phi]\begin{bmatrix}\mathbf{R}\\ \mathbf{F}\\ \mathbf{\Phi}\end{bmatrix} = 0 \tag{11}$$

where

$$\bar{\mathbf{P}}_F = \mathbf{P}_Q\mathbf{Q}_F + \mathbf{P}_F \tag{12}$$

and

$$\bar{\mathbf{P}}_\Phi = \mathbf{P}_Q\mathbf{Q}_\Phi \tag{13}$$

For subsequent analysis it is preferable to combine $\mathbf{R}$ and $\mathbf{F}$ into a single matrix

$$\bar{\mathbf{F}} = \begin{bmatrix}\mathbf{R}\\ \mathbf{F}\end{bmatrix} \tag{14}$$

so that Eq. (11) can be rewritten as

$$[\bar{\mathbf{P}}\ \bar{\mathbf{P}}_\Phi]\begin{bmatrix}\bar{\mathbf{F}}\\ \mathbf{\Phi}\end{bmatrix} = 0 \tag{15}$$

where

$$\bar{\mathbf{P}} = [\mathbf{P}_R\ \bar{\mathbf{P}}_F] \tag{16}$$

Selection of the Interaction Redundancies

If the substructures are joined together in a statically determinate manner, then the matrix $\bar{\mathbf{P}}$ is a square, nonsingular matrix and Eq. (15) can be solved directly to yield

$$\bar{\mathbf{F}} = -\bar{\mathbf{P}}^{-1}\ \bar{\mathbf{P}}_\Phi\ \mathbf{\Phi} \tag{17}$$

If, on the other hand, the substructures are connected in a redundant manner, then $\bar{\mathbf{P}}$ is a rectangular matrix with the number of columns greater than the number of rows and, consequently, no direct solution to Eq. (15) can be obtained. For such cases, the equilibrium equations are inadequate in number to determine all the substructure boundary forces and they therefore must be supplemented by the equations of displacement compatibility. To formulate the compatibility equations, we must first select from $\bar{\mathbf{F}}$ a set of redundant interaction forces and then introduce a set of structural cuts corresponding to the selected redundancies on the substructure boundaries. The cut structure then becomes a statically determinate one for the boundary forces. Naturally, each substructure by itself may be highly redundant. It therefore follows that the substructure boundary forces $\bar{\mathbf{F}}$ can be separated into interaction redundancies $\mathbf{X}$ and forces $\bar{\mathbf{F}}_0$ in the statically determinate cut structure, i.e.,

$$\bar{\mathbf{F}} = \begin{bmatrix}\bar{\mathbf{F}}_0\\ \mathbf{X}\end{bmatrix} \tag{18}$$

Using Eq. (18) the internal equilibrium equations (15) can be modified so that

$$[\mathbf{M}\ \bar{\mathbf{P}}_x\ \bar{\mathbf{P}}_\Phi]\begin{bmatrix}\bar{\mathbf{F}}_0\\ \mathbf{X}\\ \mathbf{\Phi}\end{bmatrix} = 0 \tag{19}$$

where

$$[\mathbf{M}\ \bar{\mathbf{P}}_x] = \bar{\mathbf{P}} \tag{20}$$

Since the forces $\bar{\mathbf{F}}_0$ are statically determinate, it follows therefore that $\mathbf{M}^{-1}$ exists and that the solution for $\bar{\mathbf{F}}_0$ is

$$\begin{aligned}\bar{\mathbf{F}}_0 &= -\mathbf{M}^{-1}\bar{\mathbf{P}}_x\mathbf{X} - \mathbf{M}^{-1}\bar{\mathbf{P}}_\Phi\mathbf{\Phi}\\ &= \mathbf{q}_x\mathbf{X} + \mathbf{q}_\Phi\mathbf{\Phi}\end{aligned} \tag{21}$$

where

$$\mathbf{q}_x = -\mathbf{M}^{-1}\bar{\mathbf{P}}_x \tag{22}$$

$$\mathbf{q}_\Phi = -\mathbf{M}^{-1}\bar{\mathbf{P}}_\Phi \tag{23}$$

The solution represented by Eq. (21) is obtained by the Jordanian elimination technique applied to Eq. (19). The rectangular matrix $[\bar{\mathbf{P}}\ \bar{\mathbf{P}}_\Phi]$ from Eq. (19) is premultiplied by a series of transformation matrices $\mathbf{T}_1 \ldots \mathbf{T}_n$, where n denotes the number of rows in $\bar{\mathbf{P}}$, which change the submatrix $\mathbf{M}$ from $\bar{\mathbf{P}}$ into an identity matrix $\mathbf{I}$. Symbolically, this operation may be represented by the matrix equation

$$\mathbf{T}_n \ldots \mathbf{T}_1[\bar{\mathbf{P}} \vdots \bar{\mathbf{P}}_\Phi] = [\mathbf{I} \vdots \mathbf{M}^{-1}\bar{\mathbf{P}}_x \vdots \mathbf{M}^{-1}\bar{\mathbf{P}}_\Phi] \tag{24}$$

The computer program developed for this operation, frequently referred to as the structure cutter,[4] insures, through the use of flexibility weighting factors associated with $\bar{\mathbf{F}}$, that the selection of the interaction redundancies $\mathbf{X}$ leads to the stiffest cut structure. It should be noted that the columns of $\bar{\mathbf{P}}_x$ selected from $\bar{\mathbf{P}}$ by the computer directly indicate which boundary interaction forces have been selected as the redundants.

Combining now Eq. (21) with the identity $\mathbf{X} = \mathbf{X}$ it follows that

$$\bar{\mathbf{F}} = \begin{bmatrix} \bar{\mathbf{F}}_0 \\ \mathbf{X} \end{bmatrix} = \begin{bmatrix} \mathbf{q}_x \\ \mathbf{I} \end{bmatrix}\mathbf{X} + \begin{bmatrix} \mathbf{q}_\Phi \\ 0 \end{bmatrix}\mathbf{\Phi}$$

$$= \mathbf{f}_x\mathbf{X} + \mathbf{f}_\Phi\mathbf{\Phi} \quad (25)$$

where

$$\mathbf{f}_x = \begin{bmatrix} \mathbf{q}_x \\ \mathbf{I} \end{bmatrix} \quad (26)$$

$$\mathbf{f}_\Phi = \begin{bmatrix} \mathbf{q}_\Phi \\ 0 \end{bmatrix} \quad (27)$$

represent substructure boundary forces due to unit values of interaction redundancies $\mathbf{X}$ and external loading $\mathbf{\Phi}$, respectively.

The information as to what columns are selected from $\bar{\mathbf{P}}$ for $\mathbf{M}$ and $\bar{\mathbf{P}}_x$ in the structure cutter computer program can be used to formulate two column extractor matrices $\mathbf{N}$ and $\mathbf{H}_x$ such that

$$\mathbf{M} = \bar{\mathbf{P}}\,\mathbf{N} \quad (28)$$

and

$$\bar{\mathbf{P}}_x = \bar{\mathbf{P}}\,\mathbf{H}_x \quad (29)$$

The matrices $\mathbf{N}$ and $\mathbf{H}_x$ can be expressed symbolically for the purpose of the analysis as

$$\mathbf{N} = \begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix} \quad (30)$$

$$\mathbf{H}_x = \begin{bmatrix} 0 \\ \mathbf{I} \end{bmatrix} \quad (31)$$

It should be noted, however, that in practice the identity matrices of Eqs. (30) and (31) will be interspersed (see the numerical example in the Appendix). The main purpose of introducing $\mathbf{N}$ and $\mathbf{H}_x$ is to use them to generate $\mathbf{f}_x$ and $\mathbf{f}_\Phi$ directly from $\mathbf{q}_x$ and $\mathbf{q}_\Phi$. It can be demonstrated easily that this is obtained from the equations

$$\mathbf{f}_x = \mathbf{N}\mathbf{q}_x + \mathbf{H}_x \quad (32)$$

and

$$\mathbf{f}_\Phi = \mathbf{N}\mathbf{q}_\Phi \quad (33)$$

Compatibility Equations

The relative displacements in the directions of the interaction forces $\mathbf{F}^{(r)}$ on each substructure can be expressed as

$$\mathbf{v}_F^{(r)} = \mathbf{D}_{FF}^{(r)}\,\mathbf{F}^{(r)} + \mathbf{D}_{F\Phi}^{(r)}\,\mathbf{\Phi}^{(r)} + \mathbf{e}_F^{(r)} \quad (34)$$

where $\mathbf{D}_{FF}^{(r)}$ and $\mathbf{D}_{F\Phi}^{(r)}$ denote flexibility matrices for the forces $\mathbf{F}^{(r)}$ and $\mathbf{\Phi}^{(r)}$, respectively, and $\mathbf{e}_F^{(r)}$ is a column matrix of initial displacements (e.g., due to temperature distribution). The matrices $\mathbf{D}_{FF}^{(r)}$, $\mathbf{D}_{F\Phi}^{(r)}$, and $\mathbf{e}_F^{(r)}$ are determined here with respect to the substructure datum established by the substructure reactions $\mathbf{Q}^{(r)}$. For all substructures, Eqs. (34) can be combined into a single matrix equation

$$\mathbf{v}_F = \mathbf{D}_{FF}\mathbf{F} + \mathbf{D}_{F\Phi}\mathbf{\Phi} + \mathbf{e}_F \quad (35)$$

where

$$\mathbf{v}_F = \{\mathbf{v}_F^{(1)}\ \mathbf{v}_F^{(2)}, \ldots, \mathbf{v}_F^{(N)}\} \quad (36)$$

$$\mathbf{D}_{FF} = [\mathbf{D}_{FF}^{(1)}\ \mathbf{D}_{FF}^{(2)}, \ldots, \mathbf{D}_{FF}^{(N)}] \quad (37)$$

$$\mathbf{D}_{F\Phi} = [\mathbf{D}_{F\Phi}^{(1)}\ \mathbf{D}_{F\Phi}^{(2)}, \ldots, \mathbf{D}_{F\Phi}^{(N)}] \quad (38)$$

$$\mathbf{e}_F = \{\mathbf{e}_F^{(1)}\ \mathbf{e}_F^{(2)}, \ldots, \mathbf{e}_F^{(N)}\} \quad (39)$$

For generality, it will be assumed that the joined structure reaction supports move in the directions of $\mathbf{R}$ by some specified amounts represented by the matrix $\mathbf{e}_R$. This implies that, if sinking of the supports is to be included in the analysis, the amount of sinking must be entered into $\mathbf{e}_R$ as negative values. For rigid supports $\mathbf{e}_R = 0$. The structural displacements in the directions of $\mathbf{F}$ and $\mathbf{R}$, relative to the substructure datum on each substructure, now are defined and can be used to establish the equations of compatibility of displacements on the substructure boundaries. These equations can be derived most conveniently by the application of the Unit Load Theorem[5] that states that

$$\mathbf{f}_x^T\begin{bmatrix} \mathbf{e}_R \\ \mathbf{v}_F \end{bmatrix} = 0 \quad (40)$$

or

$$\mathbf{f}_x^T\bar{\mathbf{v}}_F = 0 \quad (40a)$$

where

$$\bar{\mathbf{v}}_F = \begin{bmatrix} \mathbf{e}_R \\ \mathbf{v}_F \end{bmatrix} \quad (41)$$

Substituting Eq. (35) into (40) it follows that

$$\mathbf{f}_x^T\begin{bmatrix} 0 & 0 \\ 0 & \mathbf{D}_{FF} \end{bmatrix}\begin{bmatrix} \mathbf{R} \\ \mathbf{F} \end{bmatrix} + \mathbf{f}_x^T\begin{bmatrix} 0 \\ \mathbf{D}_{F\Phi} \end{bmatrix}\mathbf{\Phi} + \mathbf{f}_x^T\begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_F \end{bmatrix} = 0 \quad (42)$$

or

$$\mathbf{f}_x^T\bar{\mathbf{D}}_{FF}\,\bar{\mathbf{F}} + \mathbf{f}_x^T\bar{\mathbf{D}}_{F\Phi}\mathbf{\Phi} + \mathbf{f}_x^T\mathbf{e} = 0 \quad (42a)$$

where

$$\bar{\mathbf{D}}_{FF} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{D}_{FF} \end{bmatrix} \quad (43)$$

$$\bar{\mathbf{D}}_{F\Phi} = \begin{bmatrix} 0 \\ \mathbf{D}_{F\Phi} \end{bmatrix} \quad (44)$$

$$\mathbf{e} = \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_F \end{bmatrix} \quad (45)$$

Now using Eqs. (25) and (42a) the following equation for the unknown interaction redundancies $\mathbf{X}$ is obtained

$$\mathbf{f}_x^T\bar{\mathbf{D}}_{FF}\,\mathbf{f}_x\mathbf{X} + \mathbf{f}_x^T\bar{\mathbf{D}}_{FF}\,\mathbf{f}_\Phi\mathbf{\Phi} + \mathbf{f}_x^T\bar{\mathbf{D}}_{F\Phi}\mathbf{\Phi} + \mathbf{f}_x^T\mathbf{e} = 0$$

or

$$\mathbf{D}_{xx}\mathbf{X} + \mathbf{D}_{x\Phi}\mathbf{\Phi} + \mathbf{D}_{xe} = 0 \quad (46)$$

with

$$\mathbf{D}_{xx} = \mathbf{f}_x^T\,\bar{\mathbf{D}}_{FF}\,\mathbf{f}_x \quad (47)$$

$$\mathbf{D}_{x\Phi} = \mathbf{f}_x^T\,\bar{\mathbf{D}}_{FF}\,\mathbf{f}_\Phi + \mathbf{f}_x^T\,\bar{\mathbf{D}}_{F\Phi} \quad (48)$$

$$\mathbf{D}_{xe} = \mathbf{f}_x^T\mathbf{e} \quad (49)$$

Joined Structure

The solution for the redundancies $\mathbf{X}$ from Eq. (46) is

$$\mathbf{X} = -\mathbf{D}_{xx}^{-1}(\mathbf{D}_{x\Phi}\mathbf{\Phi} + \mathbf{D}_{xe}) \quad (50)$$

or

$$\mathbf{X} = \mathbf{X}_\Phi\,\mathbf{\Phi} + \mathbf{X}_e\mathbf{e} \quad (50a)$$

where

$$\mathbf{X}_\Phi = -\mathbf{D}_{xx}^{-1}\,\mathbf{D}_{x\Phi} \quad (51)$$

$$\mathbf{X}_e = -\mathbf{D}_{xx}^{-1}\,\mathbf{f}_x^T \quad (52)$$

represent the interaction redundancies due to unit values of external loads $\mathbf{\Phi}$ and initial displacements $\mathbf{e}$, respectively.

Substituting Eq. (50a) into (25) we have finally that

$$\bar{\mathbf{F}} = (\mathbf{f}_\Phi - \mathbf{f}_x\mathbf{D}_{xx}^{-1}\mathbf{D}_{x\Phi})\mathbf{\Phi} - \mathbf{f}_x\mathbf{D}_{xx}^{-1}\mathbf{D}_{xe} \quad (53)$$

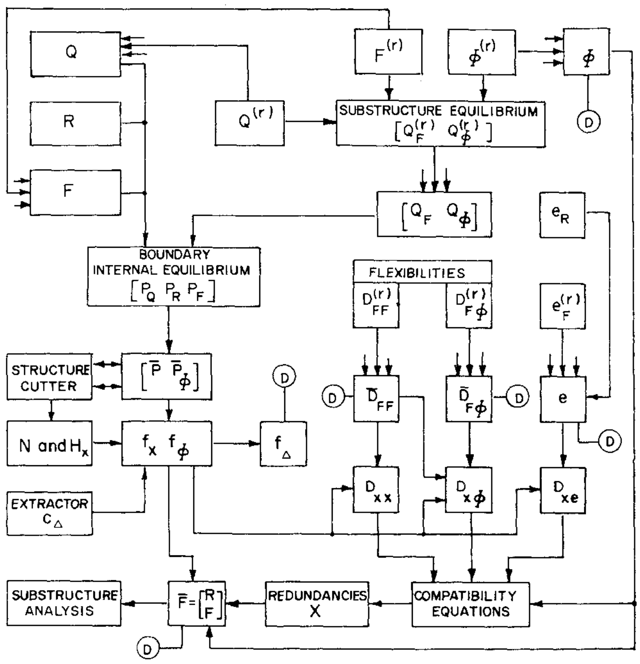

Fig. 4 Flow diagram for stress analysis.

or

$$\bar{\mathbf{F}} = \bar{\mathbf{F}}_\Phi \mathbf{\Phi} + \bar{\mathbf{F}}_e \, \mathbf{e} \tag{53a}$$

where

$$\bar{\mathbf{F}}_\Phi = \mathbf{f}_\Phi - \mathbf{f}_x \mathbf{D}_{xx}^{-1} \mathbf{D}_{x\Phi} \tag{54}$$

$$\bar{\mathbf{F}}_e = -\mathbf{f}_x \mathbf{D}_{xx}^{-1} \mathbf{f}_x^T \tag{55}$$

represent the substructure boundary forces due to unit values of external loads $\mathbf{\Phi}$ and initial displacements $\mathbf{e}$, respectively. It should be noted that the sequence of forces in $\bar{\mathbf{F}}$ is first $\mathbf{R}$ followed by $\mathbf{F}^{(1)}, \mathbf{F}^{(2)}, \ldots, \mathbf{F}^{(r)}, \ldots, \mathbf{F}^{(N)}$. Consequently, the interaction forces for any substructure can be identified easily and extracted from the computer output. These forces then are used to determine the stress distribution in the substructures, the size of which is such that small capacity stress analysis computer programs can be utilized. The stress analysis of the substructures may be carried out, of course, either by the matrix displacement or force methods. A schematic flow diagram for the complete analysis by the present method is shown in Fig. 4, where for simplicity only the main steps in the computation have been indicated. Symbols D are used there to represent data required for the deflection analysis.

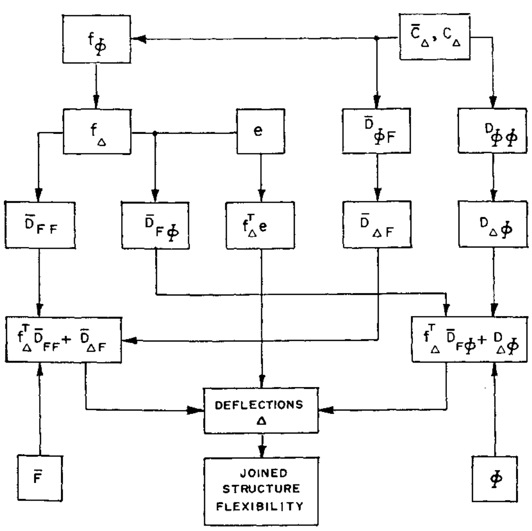

Fig. 5 Flow diagram for deflection analysis.

Deflections

To determine a deflection of the joined structure a unit load (either concentrated force or moment) is applied in the direction for which the displacement is required, and for convenience this load will be assumed to coincide with one of the external loads $\mathbf{\Phi}$. Denoting the required displacement by Δ it follows that the virtual complementary work δW^* is simply

$$\delta W^* = 1 \times \Delta \tag{56}$$

Now using Eqs. (35, 41, and 43–45), the relative displacements in the directions of $\bar{\mathbf{F}}$ caused by the external loading and initial deformations are

$$\bar{\mathbf{v}}_F = \bar{\mathbf{D}}_{FF} \bar{\mathbf{F}} + \bar{\mathbf{D}}_{F\Phi} \mathbf{\Phi} + \mathbf{e} \tag{57}$$

while the relative displacement in the direction of Δ is

$$v_\Delta = \bar{\mathbf{D}}_{\Delta F} + \mathbf{D}_{\Delta\Phi} \tag{58}$$

where $\bar{\mathbf{D}}_{\Delta F}$ and $\mathbf{D}_{\Delta\Phi}$ represent displacements due to unit values of $\bar{\mathbf{F}}$ and $\mathbf{\Phi}$, respectively. The virtual complementary work therefore can be expressed, alternatively, as

$$\delta W^* = \mathbf{f}_\Delta^T \bar{\mathbf{v}}_F + 1 \times v_\Delta \tag{59}$$

where $\mathbf{f}_\Delta$ represents the interaction forces $\bar{\mathbf{F}}$ caused by the unit load in the direction Δ.

Equating Eqs. (56) and (59) and then using Eqs. (57) and (58), it is clear that the displacements on the joined structure can be calculated from

$$\mathbf{\Delta} = (\mathbf{f}_\Delta^T \bar{\mathbf{D}}_{FF} + \bar{\mathbf{D}}_{\Delta F})\bar{\mathbf{F}} + (\mathbf{f}_\Delta^T \bar{\mathbf{D}}_{F\Phi} + \mathbf{D}_{\Delta\Phi})\mathbf{\Phi} + \mathbf{f}_\Delta^T \mathbf{e} \tag{60}$$

where $\mathbf{f}_\Delta$ represents now the forces $\bar{\mathbf{F}}$ caused by unit loads in the directions $\mathbf{\Delta}$.

Substituting Eq. (53a) into (60) it can be demonstrated that

$$\mathbf{\Delta} = \mathbf{\Delta}_\Phi \mathbf{\Phi} + \mathbf{\Delta}_e \, \mathbf{e} \tag{61}$$

where

$$\mathbf{\Delta}_\Phi = (\mathbf{f}_\Delta^T \bar{\mathbf{D}}_{FF} + \bar{\mathbf{D}}_{\Delta F})\bar{\mathbf{F}}_\Phi + \mathbf{f}_\Delta^T \bar{\mathbf{D}}_{F\Phi} + \mathbf{D}_{\Delta\Phi} \tag{62}$$

$$\mathbf{\Delta}_e = \mathbf{f}_\Delta^T + (\mathbf{f}_\Delta^T \bar{\mathbf{D}}_{FF} + \bar{\mathbf{D}}_{\Delta F})\bar{\mathbf{F}}_e \tag{63}$$

represent the joined structure displacements due to unit values of $\mathbf{\Phi}$ and $\mathbf{e}$. Eq. (62) can be used to determine the flexibility matrix for the directions of the applied loads $\mathbf{\Phi}$ if $\mathbf{f}_\Phi$ is used in place of $\mathbf{f}_\Delta$. A schematic flow diagram for the deflection analysis is shown in Fig. 5.

In practice, the matrices $\mathbf{f}_\Delta$, $\bar{\mathbf{D}}_{\Delta F}$, and $\mathbf{D}_{\Delta\Phi}$ are determined from

$$\mathbf{f}_\Delta = \mathbf{f}_\Phi \bar{\mathbf{C}}_\Delta \tag{64}$$

$$\bar{\mathbf{D}}_{\Delta F} = \mathbf{C}_\Delta \bar{\mathbf{D}}_{\Phi F} \tag{65}$$

$$\bar{\mathbf{D}}_{\Delta\Phi} = \mathbf{C}_\Delta \mathbf{D}_{\Phi\Phi} \tag{66}$$

where $\bar{\mathbf{C}}_\Delta$ and $\mathbf{C}_\Delta$ are suitable extractor matrices. The matrices $\bar{\mathbf{D}}_{FF}$, $\bar{\mathbf{D}}_{F\Phi}$, and $\mathbf{D}_{\Phi\Phi}$ are compiled from the substructure flexibility matrices of the form

$$\mathbf{D}^{(r)} = \begin{bmatrix} \mathbf{D}_{FF}^{(r)} & \mathbf{D}_{F\Phi}^{(r)} \\ \mathbf{D}_{\Phi F}^{(r)} & \mathbf{D}_{\Phi\Phi}^{(r)} \end{bmatrix} \tag{67}$$

Appendix: Analysis of a Two-Bay Truss

As an example of the new method of substructure analysis a simple two-bay pin-jointed truss (Fig. 6) will be analyzed for two different loading conditions represented by the matrix

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_1^{(1)} \\ \mathbf{\Phi}_2^{(1)} \\ \mathbf{\Phi}_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \times 10^3 \text{ lb} \tag{A1}$$

The truss is supported at one end in a statically indeterminate manner by four reactions. The reaction force complements on the joined structure are represented by the symbols $R_1 \ldots R_4$ (Fig. 7). The structure is partitioned into two substructures by disconnecting the outer bay from the remainder of the structure. An exploded view of the two substructures with all the boundary forces $\mathbf{F}$ and $\mathbf{Q}$ is shown in Fig. 7. Naturally, other choices for partitioning are possible also. For example, the center vertical member could be sliced vertically into two halves to form two substructures.

Because of the simplicity of the structure, the complete analysis has been carried out without the aid of a computer. A more complicated example that illustrates the analysis of a wing box structure using an IBM 7094 digital computer can be found in Ref. 6. The main steps in the analysis are reproduced below. The units used are pounds and inches.

Equations of External Equilibrium for the Substructures

External equilibrium equations are set up using Eq. (1).

Substructure 1

$$\mathbf{Q}^{(1)} = [\mathbf{Q}_F{}^{(1)}\ \mathbf{Q}_\Phi{}^{(1)}]\begin{bmatrix}\mathbf{F}^{(1)}\\ \mathbf{\Phi}^{(1)}\end{bmatrix} \qquad \begin{bmatrix}Q_1{}^{(1)}\\ Q_2{}^{(1)}\\ Q_3{}^{(1)}\end{bmatrix} = \left[\begin{array}{r|rrr} 0 & 0 & -1 & -1\\ -1 & 0 & -1 & 0\\ 0 & -1 & 1 & 0\end{array}\right]\begin{bmatrix}F_1{}^{(1)}\\ \Phi_1{}^{(1)}\\ \Phi_2{}^{(1)}\\ \Phi_3{}^{(1)}\end{bmatrix} \tag{A2}$$

Substructure 2

$$\mathbf{Q}^{(2)} = [\mathbf{Q}_F{}^{(2)}][\mathbf{F}^{(2)}] \qquad \begin{bmatrix}Q_1{}^{(2)}\\ Q_2{}^{(2)}\\ Q_3{}^{(2)}\end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 & -1 & 0\\ 0 & -1 & 0 & -1 & -1\\ -1 & 1 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}F_1{}^{(2)}\\ F_2{}^{(2)}\\ F_3{}^{(2)}\\ F_4{}^{(2)}\\ F_5{}^{(2)}\end{bmatrix} \tag{A3}$$

Combined Equations of External Equilibrium

It should be noted that in the present example $\mathbf{Q}_\Phi{}^{(2)}$ is not used since there are no external loads $\mathbf{\Phi}^{(2)}$ applied to the structure. The diagonal matrix $\mathbf{Q}_\Phi$ in (5) therefore is reduced to only one column of submatrices. Hence the combined equations of external equilibrium (2) become

$$\begin{bmatrix}\mathbf{Q}^{(1)}\\ \mathbf{Q}^{(2)}\end{bmatrix} = \left[\begin{array}{cc|c}\mathbf{Q}_F{}^{(1)} & \mathbf{0} & \mathbf{Q}_\Phi{}^{(1)}\\ \mathbf{0} & \mathbf{Q}_F{}^{(2)} & \mathbf{0}\end{array}\right]\left[\begin{array}{c}\mathbf{F}^{(1)}\\ \mathbf{F}^{(2)}\\ \hline \mathbf{\Phi}^{(1)}\end{array}\right] \tag{A4}$$

Equations of Internal Equilibrium

It is easy to verify, using Fig. 7, that the equations of internal equilibrium (10) are

$$\begin{array}{c} \begin{array}{cccccccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16 \end{array} \\ \left[\begin{array}{cccccc|cccc|cccccc} 0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1\\ 0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0 \end{array}\right] \end{array} \left[\begin{array}{c} Q_1{}^{(1)}\\ Q_2{}^{(1)}\\ Q_3{}^{(1)}\\ Q_1{}^{(2)}\\ Q_2{}^{(2)}\\ Q_3{}^{(2)}\\ \hline R_1\\ R_2\\ R_3\\ R_4\\ \hline F_1{}^{(1)}\\ F_1{}^{(2)}\\ F_2{}^{(2)}\\ F_3{}^{(2)}\\ F_4{}^{(2)}\\ F_5{}^{(2)} \end{array}\right] \begin{array}{c}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\end{array} = \mathbf{0} \tag{A5}$$

Equation (A5) may be described as connectivity relations defining connections between the substructures. Substituting Eq. (A4) into (A5), the following matrix equation is obtained [see Eqs. (11) or (15)]

$$\begin{array}{c} \begin{array}{ccccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13 \end{array} \\ \left[\begin{array}{rrrr|rrrrrr|rrr} 0&0&0&0&0&1&0&0&0&0&-1&1&0\\ 0&0&0&0&1&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&-1&-1\\ 0&0&0&0&-1&0&0&0&1&0&0&-1&0\\ 0&0&1&0&0&-1&1&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&1&0&0&0\\ 1&0&0&0&0&0&-1&-1&-1&0&0&0&0\\ 0&1&0&0&0&0&-1&0&-1&-1&0&0&0 \end{array}\right] \end{array} \left[\begin{array}{c} R_1\\ R_2\\ R_3\\ R_4\\ \hline F_1{}^{(1)}\\ F_1{}^{(2)}\\ F_2{}^{(2)}\\ F_3{}^{(2)}\\ F_4{}^{(2)}\\ F_5{}^{(2)}\\ \hline \Phi_1{}^{(1)}\\ \Phi_2{}^{(1)}\\ \Phi_3{}^{(1)} \end{array}\right] \begin{array}{c}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\end{array} = \mathbf{0} \tag{A6}$$

Flexibility Matrices

The flexibility matrices for the two submatrices have been calculated using the standard matrix force method of analysis. It should be noted in this connection that the substructure 1 is statically determinate while the substructure 2 has one redundancy.

$$\mathbf{D}_{FF}^{(1)} = [22] \times 10^{-6} \tag{A7}$$

$$\mathbf{D}_{F\Phi}^{(1)} = [-2 \quad 10 \quad -2] \times 10^{-6} \tag{A8}$$

$$\mathbf{D}_{\Phi\Phi}^{(1)} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 10 & 0 \\ 0 & 0 & 2 \end{bmatrix} \times 10^{-6} \tag{A9}$$

$$\mathbf{D}_{FF}^{(2)} = \begin{bmatrix} 1.8333 & & & & \text{symmetric} \\ -1.1667 & 5.8333 & & & \\ -0.1667 & 0.8333 & 1.8333 & & \\ -1.0000 & 5.0000 & 1.0000 & 6.0000 & \\ -0.1667 & 0.8333 & -0.1667 & 1.0000 & 1.8333 \end{bmatrix} \times 10^{-6} \tag{A10}$$

The flexibility matrices $\mathbf{D}_{F\Phi}^{(2)}$ and $\mathbf{D}_{\Phi\Phi}^{(2)}$ are not used since there are no external loads applied to the substructure 2.

Combined Flexibility Matrices

From Eqs. (37, 38, 43 and 44) it follows that

$$\bar{\mathbf{D}}_{FF} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{D}_{FF}^{(1)} & 0 \\ 0 & 0 & \mathbf{D}_{FF}^{(2)} \end{bmatrix} \tag{A11}$$

$$\bar{\mathbf{D}}_{F\Phi} = \begin{bmatrix} 0 \\ \mathbf{D}_{F\Phi}^{(1)} \\ 0 \end{bmatrix} = \bar{\mathbf{D}}_{\Phi F}^{T} \tag{A12}$$

Jordanian Elimination (Matrices $\mathbf{f}_x$ and $\mathbf{f}_\Phi$)

Application of the Jordanian elimination technique, represented symbolically by Eq. (24), to the equilibrium equations (A6) leads directly to the generation of the matrices $\mathbf{q}_x$ and $\mathbf{q}_\Phi$ given by Eqs. (22) and (23). A number of column interchanges were required to continue with the elimination process. The forces $F_1^{(1)}$ and $F_5^{(2)}$ were selected as the redundant boundary forces. The results obtained are shown by Eqs. (A13) and (A14) where the numbers assigned to rows and columns refer to the original column numbers in Eq. (A6). Thus the redundant boundary forces are identified by numbers 5 and 10. Only the final steps leading to Eqs. (32) and (33) are reproduced.

$$\mathbf{f}_x = \mathbf{N}\mathbf{q}_x + \mathbf{H}_x = \begin{array}{c|cccccccc} & 6 & 7 & 8 & 9 & 3 & 4 & 1 & 2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \; \begin{array}{c|cc} & & 10 \\ \hline 6 & 0 & 0 \\ 7 & -1 & 0 \\ 8 & 0 & 0 \\ 9 & 1 & 0 \\ 3 & 0 & 0 \\ 4 & 0 & -1 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{array} + \begin{array}{c|cc} & 5 & 10 \\ \hline 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \\ 4 & 0 & 0 \\ 5 & 1 & 0 \\ 6 & 0 & 0 \\ 7 & 0 & 0 \\ 8 & 0 & 0 \\ 9 & 0 & 0 \\ 10 & 0 & 1 \end{array} = \begin{array}{c|cc} & 5 & 10 \\ \hline 1 & 0 & 0 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \\ 4 & 0 & -1 \\ 5 & 1 & 0 \\ 6 & 0 & 0 \\ 7 & -1 & 0 \\ 8 & 0 & 0 \\ 9 & 1 & 0 \\ 10 & 0 & 1 \end{array} \tag{A13}$$

$$\mathbf{f}_\Phi = \mathbf{N}\mathbf{q}_\Phi = \begin{array}{c|cccccccc} & 6 & 7 & 8 & 9 & 3 & 4 & 1 & 2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \; \begin{array}{c|ccc} & 11 & 12 & 13 \\ \hline 6 & 1 & -1 & 0 \\ 7 & 0 & 0 & 0 \\ 8 & 0 & 1 & 1 \\ 9 & 0 & 1 & 0 \\ 3 & 1 & -2 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & 1 & 0 \end{array} = \begin{array}{c|ccc} & 11 & 12 & 13 \\ \hline 1 & 0 & 2 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & -2 & 0 \\ 4 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 6 & 1 & -1 & 0 \\ 7 & 0 & 0 & 0 \\ 8 & 0 & 1 & 1 \\ 9 & 0 & 1 & 0 \\ 10 & 0 & 0 & 0 \end{array} \tag{A14}$$

Compatibility Equations

Since there are no thermal strains Eq. (46) reduces to

$$\mathbf{D}_{xx}\,\mathbf{X} + \mathbf{D}_{x\Phi}\,\mathbf{\Phi} = \mathbf{0} \tag{A15}$$

where

$$\mathbf{D}_{xx} = \mathbf{f}_x^{T}\,\bar{\mathbf{D}}_{FF}\,\mathbf{f}_x = \begin{bmatrix} 23.8333 & 0.1667 \\ 0.1667 & 1.8333 \end{bmatrix} \times 10^{-6} \tag{A16}$$

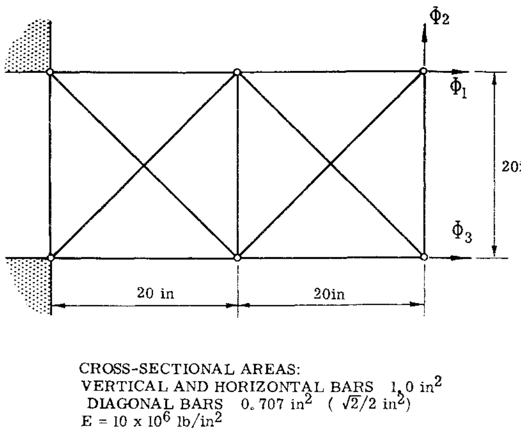

Fig. 6 Truss geometry and loading.

$$\mathbf{D}_{x\Phi} = \mathbf{f}_x^T\,\bar{\mathbf{D}}_{FF}\,\mathbf{f}_\Phi + \mathbf{f}_x^T\,\bar{\mathbf{D}}_{F\Phi} \tag{A17}$$

$$= \begin{bmatrix} 1.8333 & 11.0000 & 1.8333 \\ -0.1667 & 1.0000 & -0.1667 \end{bmatrix} \times 10^{-6}$$

Hence

$$\mathbf{X} = -\mathbf{D}_{xx}^{-1}\,\mathbf{D}_{x\Phi}\,\mathbf{\Phi} = \mathbf{X}_\Phi\,\mathbf{\Phi}$$

$$= \begin{bmatrix} 0.07634 & -0.45802 & 0.07634 \\ 0.08397 & -0.50382 & 0.08397 \end{bmatrix} \mathbf{\Phi} \tag{A18}$$

Substructure Boundary Forces

The boundary forces **R** and **F** due to unit values of the external loads are calculated from Eqs. (51) and (54). Hence

$$\bar{\mathbf{F}}_\Phi = \mathbf{f}_\Phi - \mathbf{f}_x\mathbf{D}_{xx}^{-1}\,\mathbf{D}_{x\Phi} = \mathbf{f}_\Phi + \mathbf{f}_x\,\mathbf{X}_\Phi$$

$$\bar{\mathbf{F}}_\Phi = \begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array}\begin{bmatrix} 0 & 2.0 & 1.0 \\ 0.08397 & 0.49618 & 0.08397 \\ 1.0 & -2.0 & 0 \\ -0.08397 & 0.50382 & -0.08397 \\ \hdashline 0.07634 & -0.45802 & 0.07634 \\ 1.0 & -1.0 & 0 \\ -0.07634 & 0.45802 & -0.07634 \\ 0 & 1.0 & 1.0 \\ 0.07634 & 0.54198 & 0.07634 \\ 0.08379 & -0.50382 & 0.08397 \end{bmatrix} \begin{array}{l} \left.\begin{matrix}\\ \\ \\ \\ \end{matrix}\right\} \mathbf{R} \\ \left.\begin{matrix}\\ \\ \\ \\ \\ \\ \end{matrix}\right\} \mathbf{F} \end{array} \tag{A19}$$

The actual boundary forces due to the applied loading **Φ** are given then by

$$\bar{\mathbf{F}} = \bar{\mathbf{F}}_\Phi\,\mathbf{\Phi}$$

$$\bar{\mathbf{F}} = \begin{array}{r} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array}\begin{bmatrix} 2.0 & 1.0 \\ 0.49618 & 0 \\ -2.0 & -1.0 \\ 0.50382 & 0 \\ \hdashline -0.45802 & 0 \\ -1.0 & -1.0 \\ 0.45802 & 0 \\ 1.0 & 1.0 \\ 0.54198 & 0 \\ -0.50382 & 0 \end{bmatrix} \times 10^3 \tag{A20}$$

The boundary forces **F** can be applied to each individual substructure to find stress distribution within the substructures.

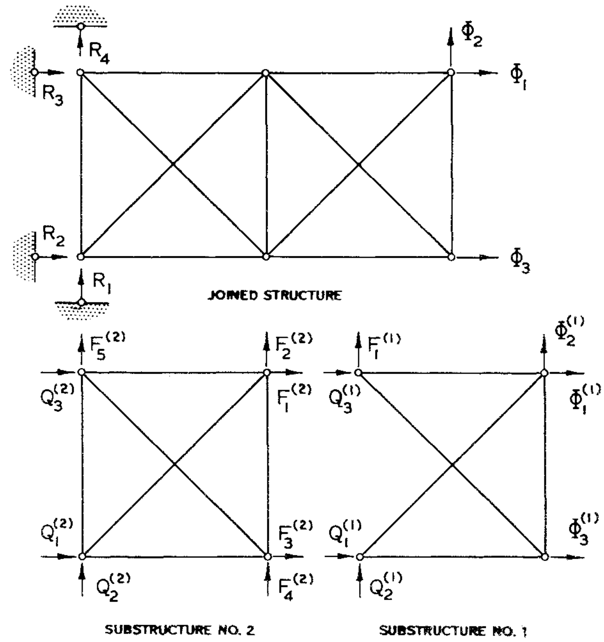

Fig. 7. Substructure boundary forces.

Deflections

To determine deflections in the direction of the applied forces **Φ** we may use $\mathbf{f}_\Delta = \mathbf{f}_\Phi$. Hence from Eq. (62) deflections due to unit values of **Φ** are given by

$$\mathbf{\Delta}_\Phi = (\mathbf{f}_\Phi^T\,\bar{\mathbf{D}}_{FF} + \bar{\mathbf{D}}_{\Phi F})\bar{\mathbf{F}}_\Phi + \mathbf{f}_\Phi^T\,\bar{\mathbf{D}}_{F\Phi} + \mathbf{D}_{\Phi\Phi} \tag{A21}$$

where in the present example

$$\mathbf{D}_{\Phi\Phi} = \mathbf{D}_{\Phi\Phi}^{(1)} \tag{A22}$$

Using previously calculated matrices the following results are obtained

$$\mathbf{\Delta}_\Phi = \begin{bmatrix} 3.679 & -4.076 & -0.3206 \\ -4.076 & 18.46 & 3.924 \\ -0.3206 & 3.924 & 3.679 \end{bmatrix} \times 10^{-6} \tag{A23}$$

$$\mathbf{\Delta} = \mathbf{\Delta}_\Phi\,\mathbf{\Phi} = \begin{bmatrix} -4.076 & -4.000 \\ 18.46 & 8.000 \\ 3.924 & 4.000 \end{bmatrix} \times 10^{-3} \tag{A24}$$

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